On the Co-existence of Primary and Secondary Transmitters in a Broadcast Network

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ABSTRACT

A secondary transmitter trying to make use of the spectrum assigned to a broadcasting service needs to be carefully planned so as to avoid interfering the primary receivers. In particular, the overlay cognitive radio paradigm postulates the use of a fraction of the secondary transmitter power to reinforce the primary signal, thus compensating for the degradation coming from the remainder of the secondary signal. Given that performance is not simply a function of the signal to noise ratio, we will analyze how to properly modify transmission strategies to avoid potential degradations in the primary service. As a study case the paper will focus on DVB-T single frequency networks, widely used in many countries worldwide for digital television broadcasting.

Categories and Subject Descriptors

C.3 [**Computer Systems Organization**]: Special purpose and application-based systems—*Signal processing systems*

General Terms

Design, Theory

Keywords

Cognitive Radio, Overlay, Broadcast, Single Frequency Network

1. INTRODUCTION

Recently, there has been an increased interest for learning the potential of those cognitive radio systems where the secondary transmitter has knowledge of the primary message [1], in what is known as the *overlay paradigm* [2]. Not surprisingly, this knowledge is possible in some practical cases like those broadcasting systems working as a Single Frequency Network (SFN), e.g., the European DVB-T based service, deployed in many countries worldwide. The primary signal is sent via satellite to some major transmitters,

which need to apply the corresponding delay to keep the synchronization required by the SFN mode[1]. Thus, a potential secondary transmitter might also gain access to the primary signal, keeping time and frequency synchronization with the primary transmitters and, therefore, join the primary network. This strategy is similar to a relay trying to reinforce a primary user, with the difference that the secondary transmitter is not really *relaying* the primary signal, but joining the SFN as any other primary transmitter. The ultimate goal is to overlay a secondary signal on the primary signal which can be decoded by secondary receivers, while preserving and possibly enforcing the quality of service (QoS) of the primary network (see Figure 1) without any modification on the primary receivers. Although following a different principle, this idea has been developed in [3], where by resorting to game theory principles, the primary transmitter adapts its power to the overlayed secondary use of the spectrum to keep its QoS. Here we will consider that the transmit power of the primary is fixed, as usual in current broadcast networks. And although the case of secondary transmitters with knowledge of the primary signal has been addressed from an information theoretic point of view, see e.g. [4] among others, there is still an important gap between *capacity-achieving* models and practical implementations where spectrum reuse is successfully achieved. In short, some of the main issues to address are:

- In broadcasting scenarios, *coverage areas* become the relevant metric as a result of the achieved *bit error rates*.

- Even in the absence of a secondary information signal, the simple transmission of the primary signal from a secondary transmitter will not necessarily improve the primary service quality, since echoes can degrade performance as it is well-known in current SFN deployments [5], unless proper countermeasures can be taken.

In practical cases, the degradation coming from the secondary echo could be higher than the power gain due to the extra contribution of the secondary transmitter. This fact was shown empirically for the deployment of a DVB-H network in recent work [6]. This type of problems is expected to be mitigated in the future with systems such as DVB-T2 [7]. This new digital terrestrial broadcasting standard includes specific schemes such as Alamouti space-time coding or constellation rotation. On the other side, some specific

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[1]There is also a format change which is not relevant for this work.

channels, such as Rayleigh fading channels, benefit from the diversity created by SFN deployments, as illustrated in [5].

Given the widespread current use of DVB-T, we will focus on this multicarrier technology as support for the primary signal, and show how an appropriate secondary transmission of the primary signal can reinforce the original QoS, as a first step towards a cognitive secondary transmitter which additionally includes a secondary information signal.

The structure of the document is as follows: in Section 2 we will introduce the notation and the analytical expressions to be used afterwards. Section 3 explains the problems arising due to the presence of a replica sent by the secondary transmitter. In Section 4 we propose a method to overcome the channel degradation problem. Finally, in Section 5 we will derive analytical bounds for an AWGN channel.
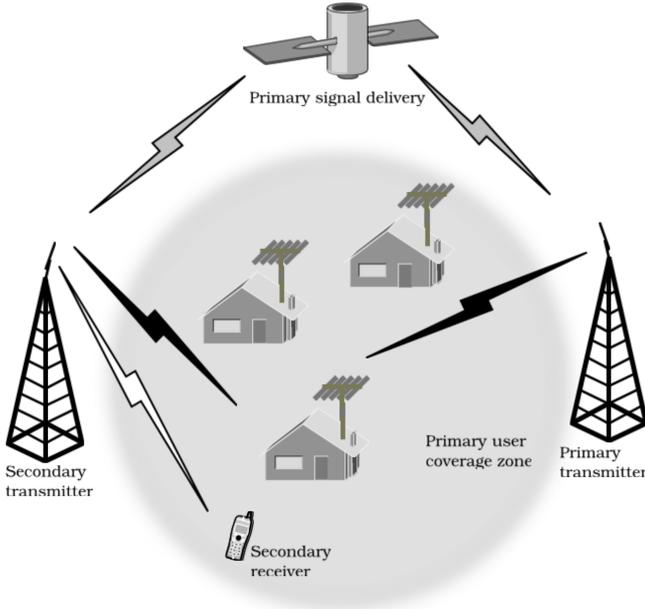

Figure 1: The secondary user transmits the primary signal, which is delivered via satellite.

2. PROBLEM STATEMENT

Throughout the paper we will use the following notation:

- $s^{(n)}[k]$ is the symbol in the kth carrier of the nth OFDM block.

- $H_i[k] = |H_i[k]| e^{j\theta_i(k)}$ is the channel coefficient of the kth carrier from the ith transmitter, $i = 1$ being the primary and $i = 2$ the secondary, to a given receiver. We will assume time-invariant channels, or at least slow-varying ones, so no index n is used for simplicity.

- $w^{(n)}[k]$ is a sample of circular symmetric white gaussian noise of variance σ^2, $w \sim \mathcal{CN}(0, \sigma^2)$.

- γ^2 is the relative power of the secondary transmitter with respect to the primary one.

- $r^{(n)}[k]$ is the received sample in the kth carrier of the nth OFDM block.

- $E_X\{\cdot\}$ will denote the expected value with respect to the distribution of the random variable X.

- N will denote the number of data carriers. For the sake of simplicity, signaling and pilot carriers are not taken into account.

For the sake of simplicity, we will assume perfect channel estimation and frequency synchronization, and an overall channel length shorter than the CP. In a first approach, we will consider a 4-QAM constellation, as the derived analytical bounds are easier to deal with, although results can be extended to higher-order constellations. In consequence, the received signal at a given location can be modeled as follows, for $k = 0, \ldots, N-1$:

$$r^{(n)}[k] = s^{(n)}[k] \times$$
$$\left(|H_1[k]| e^{j\theta_1(k)} + \gamma |H_2[k]| e^{j\theta_2(k)} \right) + w^{(n)}[k]. \quad (1)$$

Without loss of generality, we will consider $\theta_1(k) = 0$, with $\theta_2(k) = \theta(k)$ accounting for the phase difference:

$$r^{(n)}[k] = s^{(n)}[k] \left(|H_1[k]| + \gamma |H_2[k]| e^{j\theta(k)} \right) + w^{(n)}[k]. \quad (2)$$

The BER performance will by analyzed by using the following analytical bound for the BER after Viterbi, taken from [8]:

$$P_b \leq \frac{1}{2} \sum_{d=d_{min}}^{\infty} c_d P_d, \quad (3)$$

$$P_d \leq \frac{1}{2} \left[E_H \left\{ exp \left(-\frac{d_E^2 |H|^2}{4\sigma^2} \right) \right\} \right]^d \quad (4)$$

with d_{min} the minimum Hamming distance of the convolutional code, P_d the pairwise error probability (PEP) of any error event af Hamming distance d from the all-zero path, c_d the total input weight due to that error event and d_E the Euclidean distance between symbols. This bound will be instrumental to get analytical results of potential interest for practical designs.

3. CHANNEL QUALITY VS POWER

With the introduction of a secondary transmitter conveying the primary signal the Carrier-to-Noise Ratio (CNR) of primary receivers is expected to grow, at least in the absence of a secondary information signal. Unfortunately, the presence of an echo can degrade the performance of the decoder, so the CNR increase does not translate immediately into better performance. This can be confirmed keeping in mind the bound (3-4), as shown next.

From (2), the squared magnitude of kth carrier channel is given by

$$|H[k]|^2 = \left| |H_1[k]| + \gamma |H_2[k]| e^{j\theta(k)} \right|^2 = \quad (5)$$
$$|H_1[k]|^2 + \gamma^2 |H_2[k]|^2 - 2|H_1[k] H_2[k]| \gamma cos(\theta(k)).$$

$\theta(k)$ will be modeled as a uniform random variable in the interval $[0, 2\pi)$. Any correlation across carriers is not relevant, since the bound (3-4) assumes ideal frequency interleaving. Even though the expected value of (5) gets larger with γ^2,

$$E\{|H[k]|^2\} = |H_1[k]|^2 + \gamma^2 |H_2[k]|^2 > |H_1[k]|^2 \quad (6)$$

the bound (4) is exponential and a strictly convex function of (5), so

$$f\left(E_{|H|^2}\left\{|H|^2\right\}\right) \leq E_{|H|^2}\left\{f\left(|H|^2\right)\right\} \qquad (7)$$

where $f(|H|^2) = exp\left(-\frac{|H|^2 d_e^2}{4\sigma^2}\right)$ is the argument of the expected value in the PEP bound. Thus, performance can get worse as illustrated in Figure 2, which shows the result of simulating an echo over an AWGN channel for different CNR values[2]. Noticeably, BER after Viterbi decoder is degraded for a significant range of CNR values, especially those for which original BER was below $2 \cdot 10^{-4}$, the considered *quasi error free* threshold.
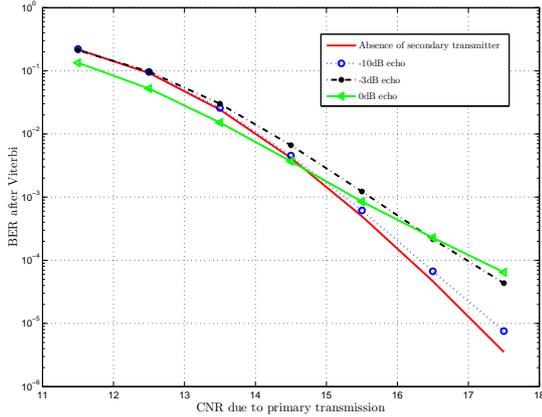

Figure 2: Convolutional rate 2/3, 8K mode, 64-QAM. The echo delay is 50% **the cyclic prefix length.**

4.   CHANNEL PRE-EQUALIZATION

Keeping in mind the broadcast primary service, and the need to preserve its quality by preventing the degradation exposed in the previous sections, we can resort to pre-equalize the primary signal coming out of the secondary transmitter. Since a linear filter would increase the effective length of the channel and, in consequence, would increase its probability of getting larger than the cyclic prefix, a circular filtering is more appropriate. Figure 3 shows the corresponding frequency weighting $F[k]$ at kth carrier, possibly non-uniform as we will see next. If we incorporate $F[k]$ in (2), then the
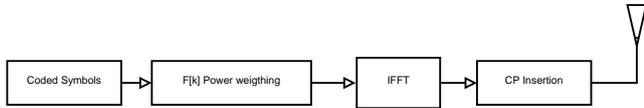

Figure 3: Pre-filtering at secondary transmitter.

post-FFT signal at reception can be written as

$$r^{(n)}[k] = s^{(n)}[k]\left(H_1[k] + \gamma F[k]H_2[k]e^{j\theta(k)}\right) + w^{(n)}[k] \quad (8)$$

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[2]In the figure, the x axis is labeled as *CNR due to primary transmission*, this is, a CNR of 18dB that is the result of two contributions of 15dB will be labeled as 15dB on the x axis, and corresponding to the curve of 0-dB echo

which, by denoting $\gamma_k \doteq \gamma F[k]$, reads as

$$r^{(n)}[k] = s^{(n)}[k]\left(H_1[k] + \gamma_k H_2[k]e^{j\theta(k)}\right) + w^{(n)}[k]. \quad (9)$$

At this point, we have all the elements to formulate an optimization problem which leads to the power distribution γ_k which minimizes the BER, subject to the power restriction $\frac{1}{N}\sum_{k=0}^{N-1}\gamma_k^2 \leq \gamma^2$. The AWGN case will be chosen for illustration in the next section.

5.   AWGN CHANNEL

If both channels H_1 and H_2 can be modeled as AWGN, then the received signal in (9) is given by

$$r^{(n)}[k] = s^{(n)}[k]\left(1 + \gamma_k e^{j\theta(k)}\right) + w^{(n)}[k]$$
$$\doteq s^{(n)}[k]H[k] + w^{(n)}[k] \qquad (10)$$

with

$$|H[k]|^2 = 1 + \gamma_k^2 - 2\gamma_k cos\left(\theta(k)\right). \qquad (11)$$

Now, assuming a 4-QAM constellation, we can rewrite the expression of the PEP (4) by applying the partition property for the expected value with respect to the probability P_k of a bit to be transmitted on the k-th carrier:

$$P_d \leq \frac{1}{2}\left[\sum_{k=0}^{N-1}E_{|H[k]|^2}\left\{e^{-\frac{d_E^2|H[k]|^2}{4\sigma^2}}\right\}P_k\right]^d = \qquad (12)$$
$$\frac{1}{2N^d}\left[\sum_{k=0}^{N-1}E_{|H[k]|^2}\left\{e^{-\frac{d_E^2|H[k]|^2}{4\sigma^2}}\right\}\right]^d$$

where $P_k = \frac{1}{N}$, as ideal interleaving among the N data carriers is assumed. Expression (12), as shown in Appendix A, boils down to

$$P_d \leq \frac{1}{2N^d}\left[e^{-\beta}\sum_{k=0}^{N-1}e^{-\beta\gamma_k^2}I_0(2\beta\gamma_k)\right]^d \qquad (13)$$

where $I_0(\cdot)$ is the modified Bessel function of the first kind of order zero, and $\beta \doteq d_e^2/(4\sigma^2)$. As the Signal-to-Noise Ratio for a QPSK constellation is $SNR = d_e^2/(2\sigma^2)$, we can write $\beta = SNR/2$. With this, we can formulate the following optimization problem:

$$\begin{array}{ll} \text{minimize} & \sum_{k=0}^{N-1}e^{-\beta\gamma_k^2}I_0(2\beta\gamma_k) \\ \text{subject to} & \frac{1}{N}\sum_{k=0}^{N-1}\gamma_k^2 \leq \gamma^2. \end{array} \qquad (14)$$

This is a non-convex problem over N variables, which makes numerical methods difficult to apply. However, if we observe that the channel degradation is caused by the term $\gamma_k^2 - 2\gamma_k cos\left(\theta(k)\right)$ being negative, then we can enforce non-negative values by means of the following power distribution[3]:

$$\gamma_k = \begin{cases} 2 & \text{if } k < \lfloor N\frac{\gamma^2}{4}\rfloor \\ 0 & \text{if } k \geq \lfloor N\frac{\gamma^2}{4}\rfloor. \end{cases} \qquad (15)$$

This power distribution, which fulfills the restriction $\frac{1}{N}\sum_{k=0}^{N-1}\gamma_k^2 \leq \gamma^2$, guarantees that no carrier suffers a received power reduction. Note that the objective is not to maximize the

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[3]We are assuming to know the received power for a given primary receiver. However, in a broadcast system this is not a reasonable approach, so the worst case must be considered to determine the power distribution.

throughput, (or minimize the BER) according to the channel state (as in the *waterfilling* power distribution), because the secondary transmitter is not aware of the quality of the different carriers, and, in any case, the result of that optimization would be different for every primary receiver. Even when different primary receivers are co-located and, therefore, the parameters β and γ are expected to be similar, a different delay between echoes or a difference in the phase will cause the resulting channel to be different. As shown in Appendix C, those points of the form $\boldsymbol{\gamma} = [\mathbf{0}_{N(1-\phi)} \; K\mathbf{1}_{N\phi}]$ with a value of K such that the power constraint is met with equality are critical points of the Lagrangian of the proposed optimization problem. Note that the number of null carriers $N(1 - \phi)$ can be zero depending on the power restriction. For this type of solutions, the optimization problem (14) can be recast as

$$\begin{aligned} \text{minimize} \quad & (1 - \phi) + \phi e^{-\beta\gamma^2/\phi} I_0(2\beta\gamma/\sqrt{\phi}) \\ \text{subject to} \quad & 0 \leq \phi \leq 1 \end{aligned} \quad (16)$$

where ϕ is the fraction of active carriers. Here, we are assuming that the number of carriers is large enough to approximate the fraction of active carriers by any real number in the interval $[0, 1]$. If the resulting optimum value of ϕ is such that $N\phi$ is not an integer, the loss of performance taking $\lfloor N\phi \rfloor$ as the number of active carriers will be negligible.

The heuristic solution (15) corresponds to $\phi = \frac{\gamma^2}{4}$, which is indeed the asymptotical solution of the optimization problem for large values of β, as shown in Appendix B. Note that the solution is only dependant on the fraction of active carriers, and not on its particular position, due to the symmetry of the problem. However, having all the active carriers together is expected to help the primary receiver in tasks such as the channel estimation.

In any case, this unidimensional optimization problem is computationally tractable as opposed to (14). We have evaluated the BER in (4) for this particular type of solutions: Figure 4 shows that the proposed method always improves the BER performance of the system, even when the *unfiltered* approach leads to a huge degradation. Moreover, the rule of thumb $\phi = \frac{\gamma^2}{4}$ is quite a good approximation to the optimum value of the fraction of active carriers, specially for the higher SNR case. SNR_0 denotes the SNR that the bound predicts for the QEF threshold, which is $SNR_0 \approx 5.6dB$.

In Figure 5 we show the value of the fraction of active carriers obtained by using Matlab function `fminbnd`; for large values of SNR (and, therefore, large values of β), we have that ϕ approaches $\gamma^2/4$ as expected.

6. FINAL REMARKS AND CONCLUSIONS

Analytical results have been provided for weighting the power distribution along carriers in a secondary transmitter which tries to enforce the primary service. This is possible provided that the primary signal is sent by satellite to the primary transmitters operating in an SFN mode, as occurs in DVB-T broadcasting, and by means of analytical bounds for the BER performance after Viterbi. When dealing with a number of receivers with different reception conditions, a trade-off will be required for the design of the power weighting, since optimality is not possible simultaneously. However, by prioritizing those receivers with lower SNR margin, the goal is still to keep the original coverage area, if not en-
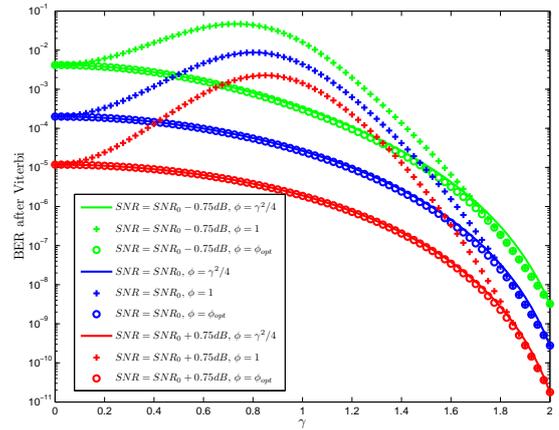

Figure 4: BER for QPSK, convolutional rate 2/3, with different secondary transmission approaches: no filtering ($\phi = 1$, filtered with $\phi = \gamma^2/4$ like in (15) and with $\phi = \phi_{opt}$ as found by `fminbnd`.

large it. Channel estimation is also of practical importance. Although ideal in this work, it has also been object of study, especially considering the type of binary solutions obtained for the power distribution. Appropriate smoothing can be performed on the achieved optimal solutions, while keeping the performance, to avoid large errors coming from the channel estimation stage. This is a first step towards the overlay reuse of the spectrum by a secondary signal. The knowledge of the primary signal needs also to be exploited somehow to add a secondary information signal, which is currently under investigation.

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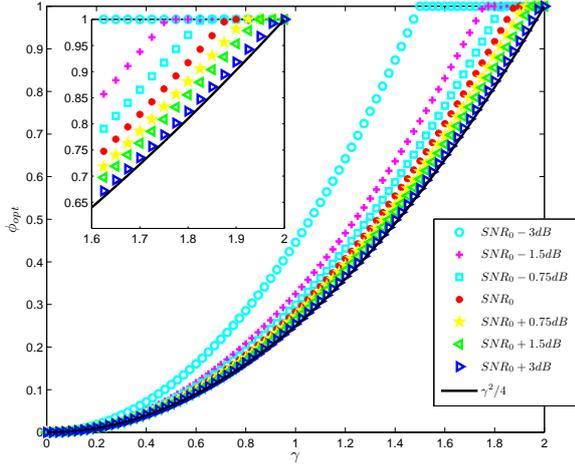
Figure 5: Optimum fraction of active carriers for different γ and SNR, obtained with `fminbnd`.

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APPENDIX

A. DERIVATION OF THE PEP FOR THE AWGN CHANNEL

We want to calculate

$$E_{|H|^2}\left\{exp(-\beta|H|^2)\right\}. \tag{17}$$

From expression (11), and assuming $\theta \sim U(0, 2\pi]$,

$$E_{|H|^2}\left\{exp(-\beta|H|^2)\right\} = E_\theta\left\{e^{-\beta(1+\gamma^2-2\gamma cos(\theta))}\right\} = \tag{18}$$
$$e^{-\beta}e^{-\beta\gamma^2}E_\theta\left\{e^{2\beta\gamma cos(\theta))}\right\}$$

where we dropped the index $[k]$ for the sake of simplicity. Noticing that

$$E_\theta\left\{e^{2\beta\gamma cos(\theta))}\right\} = \frac{1}{2\pi}\int_0^{2\pi} e^{2\beta\gamma cos(\theta)}\mathrm{d}\theta = I_0(2\beta\gamma), \tag{19}$$

we have that

$$E_{|H|^2}\left\{exp(-\beta|H|^2)\right\} = e^{-\beta}e^{-\beta\gamma^2}I_0(2\beta\gamma). \tag{20}$$

B. ASYMPTOTIC OPTIMUM VALUE FOR ϕ

We start with the function to minimize in (16):

$$f(\phi) = (1-\phi) + \phi e^{-\beta\gamma^2/\phi}I_0(2\beta\gamma/\sqrt{\phi}). \tag{21}$$

In order to find a minimum of this function, we take its derivative

$$\frac{d}{d\phi}\left(f(\phi)\right) =$$
$$e^{-\beta\gamma^2/\phi}\left(I_0(2\beta\gamma/\sqrt{\phi})\left(1+\frac{\beta\gamma^2}{\phi}\right) - I_1(2\beta\gamma/\sqrt{\phi})\frac{\beta\gamma}{\sqrt{\phi}}\right) - 1. \tag{22}$$

If we use the asymptotic approximation for the Bessel function $I_{0,1}(x) \approx \frac{e^x}{\sqrt{2\pi x}}$, and make the variable change $\alpha = \frac{\gamma}{\sqrt{\phi}}$, after equating (22) to zero we have

$$e^{2\beta\alpha}\left(\frac{1-\beta\alpha+\beta\alpha^2}{\sqrt{4\pi\beta\alpha}}\right) = e^{\beta\alpha^2}. \tag{23}$$

For asymptotically large β, the expression between parenthesis can be ignored, so the remaining expression is

$$e^{2\beta\alpha} = e^{\beta\alpha^2}. \tag{24}$$

Therefore, we have $\alpha = 2$, which leads to a value of $\phi = \frac{\gamma^2}{4}$. Note that this expression is only valid for values of $\gamma < 2$. In fact, if $\gamma > 2$, the solution of the problem is to transmit over all carriers with equal power, i.e., $\phi = 1$.

C. OPTIMALITY CONDITIONS FOR THE OPTIMIZATION PROBLEM

The proposed optimization problem is

$$\begin{array}{ll} \text{minimize} & \sum_{k=0}^{N-1} e^{-\beta\gamma_k^2}I_0(2\beta\gamma_k) \\ \text{subject to} & \frac{1}{N}\sum_{k=0}^{N-1}\gamma_k^2 \le \gamma^2. \end{array} \tag{25}$$

C.1 KKT Conditions

The Lagrangian of the optimization problem is

$$L(\boldsymbol{\gamma}, \lambda) = \sum_{k=0}^{N-1} e^{-\beta\gamma_k^2}I_0(2\beta\gamma_k) + \lambda\left(\sum_{k=0}^{N-1}\gamma_k^2 - N\gamma^2\right). \tag{26}$$

Since the gradient of the objective function is

$$(\nabla f_0(\boldsymbol{\gamma}))_k = 2e^{-\beta\gamma_k^2}\left(-\beta\gamma_k I_0(2\beta\gamma_k) + \beta I_1(2\beta\gamma_k)\right), \tag{27}$$

the resulting KKT conditions (necessary conditions for a point to be optimal) are

$$2e^{-\beta\gamma_k^2}\left(-\beta\gamma_k I_0(2\beta\gamma_k) + \beta I_1(2\beta\gamma_k)\right) + 2\lambda\gamma_k = 0 \ \forall k, \tag{28}$$

$$\lambda\left(\sum_{k=0}^{N-1}\gamma_k^2 - N\gamma^2\right) = 0, \tag{29}$$

$$\lambda \ge 0. \tag{30}$$

We will distinguish two cases: when $\sum_{k=0}^{N-1}\gamma_k^2 - N\gamma^2 = 0$, so λ is not forced to be zero (we will refer to the constraint as *active* in that case), and when $\sum_{k=0}^{N-1}\gamma_k^2 - N\gamma^2 < 0$, so λ is forced to be zero in order to meet condition (29).

C.1.1 Non-active constraint

In this case, we have $\lambda = 0$, so the resulting condition is

$$2e^{-\beta\gamma_k^2}\left(-\beta\gamma_k I_0(2\beta\gamma_k) + \beta I_1(2\beta\gamma_k)\right) = 0 \; \forall k \qquad (31)$$

or, equivalently

$$\gamma_k I_0(2\beta\gamma_k) = I_1(2\beta\gamma_k) \; \forall k. \qquad (32)$$

PROPOSITION C.1. *The nontrivial solutions for (32) are in the interval* $\sqrt{\frac{\beta-1}{\beta}} < \gamma_k \le 1$ *for* $\beta > 1$. *For* $\beta \le 1$, *the only solution is* $\gamma_k = 0$.

PROOF. We will assume $\gamma_k \ne 0$. Using expression (48), we can write

$$I_0(2\beta\gamma_k) \ge \frac{1}{\beta\gamma_k} I_1(2\beta\gamma_k). \qquad (33)$$

Combining this expression with (32) we obtain

$$I_0(2\beta\gamma_k) \ge \frac{1}{\beta} I_0(2\beta\gamma_k), \qquad (34)$$

so $\beta \ge 1$.

The proof of $\gamma_k \le 1$ is straightforward. As $I_0(x) \ge I_1(x)$, using equation (32) we obtain the desired result. The proof of $\gamma_k > \sqrt{\frac{\beta-1}{\beta}}$ is more involved. Starting with expression (49), we have that

$$I_1^2(2\beta\gamma_k) > I_0(2\beta\gamma_k)I_2(2\beta\gamma_k), \qquad (35)$$

expression that, combined with (48) leads to

$$\gamma_k^2 I_0^2(2\beta\gamma_k) > I_0(2\beta\gamma_k)I_2(2\beta\gamma_k). \qquad (36)$$

Finally, combining equation (47) and (32), we have that

$$I_2(2\beta\gamma_k) = \left(1 - \frac{1}{\beta}\right)I_0(2\beta\gamma_k), \qquad (37)$$

so (36) reads as

$$\gamma_k^2 I_0^2(2\beta\gamma_k) > \left(1 - \frac{1}{\beta}\right)I_0^2(2\beta\gamma_k), \qquad (38)$$

or, equivalently

$$\gamma_k > \sqrt{\frac{\beta-1}{\beta}}. \qquad (39)$$

$\square$

PROPOSITION C.2. *The nontrivial solutions for (32) are not local minima of the optimization problem.*

PROOF. In order to be a local minima, the Hessian matrix of the objective function has to be positive definite. The Hessian is a diagonal matrix with elements

$$(L(\boldsymbol{\gamma}))_{k,k} = L(\gamma_k) = \qquad (40)$$
$$2e^{-\beta\gamma_k^2}\left(-4\beta^2\gamma_k I_1(2\beta\gamma_k) + \right.$$
$$\left.\beta^2 I_2(2\beta\gamma_k) + \left(2\beta\gamma_k^2 + \beta - 1\right)I_0(2\beta\gamma_k)\right).$$

Moreover, we have that

$$-4\beta\gamma_k I_1(2\beta\gamma_k) + \beta I_2(2\beta\gamma_k) + \qquad (41)$$
$$\beta\left(2\beta\gamma_k^2 + \beta - 1\right)I_0(2\beta\gamma_k) \overset{(i)}{=}$$
$$(-2\beta\gamma_k^2 - 1 + b)I_0(2\beta\gamma_k) + \beta I_0(2\beta\gamma_k) \overset{(ii)}{=}$$
$$(-2\beta\gamma^2 + 2\beta - 2)I_0(2\beta\gamma_k),$$

where (i) is given by (32) and (ii) by (47) and (32).

As all the elements must be positive if $\boldsymbol{\gamma}$ is a local minimum, and I_0 is strictly positive, the condition for the minimum is $\gamma_k < \sqrt{\frac{\beta-1}{\beta}}$, which contradicts proposition C.1. $\square$

PROPOSITION C.3. *If* $\beta < 1$, $\boldsymbol{0}$ *is a local maximizer for the optimization problem. If* $\beta > 1$, $\boldsymbol{0}$ *is a local minimizer for the optimization problem.*

PROOF. From the expression (40)

$$L(0) = \beta - 1, \qquad (42)$$

So the Hessian will be positive definite if $\beta > 1$, and negative definite if $\beta < 1$. $\square$

C.1.2 Active constraint

In this case, we have the following necessary conditions for the point $\boldsymbol{\gamma}$ to be optimal

$$-2\beta\gamma_k e^{-\beta\gamma_k^2}I_0(2\beta\gamma_k) + 2\beta I_1(2\beta\gamma_k)e^{-\beta\gamma_k^2}, \qquad (43)$$
$$+2\lambda\gamma_k = 0 \; \forall i = 1, ..., N,$$
$$\lambda \ge 0. \qquad (44)$$

The condition is met if $\gamma_k^2 = 0$, as $I_1(0) = 0$. If $\gamma_k^2 \ne 0$, we can rewrite (43) as

$$\lambda = \beta e^{-\beta\gamma_k^2}\left(I_0(2\beta\gamma_k) - \frac{1}{\gamma_k}I_1(2\beta\gamma_k)\right) \forall \, k, \qquad (45)$$

so it can be seen that points of the form $\boldsymbol{\gamma}_M = [\boldsymbol{0}_{N-M} \; k\boldsymbol{1}_M]$ where the power constraint is active are critical points of the Lagrangian. In fact, due to the symmetry of the optimization problem, any permutation of the elements in $\boldsymbol{\gamma}_M$ will lead to the same value of the objective function, so this family of critical points can be written as $\mathcal{O} = \{\text{Perm}\{\boldsymbol{\gamma}_M\}, M = 0, .., N\}$.

As the function

$$\lambda(\gamma_k) = \beta e^{-\beta\gamma_k^2}\left(I_0(2\beta\gamma_k) - \frac{1}{\gamma_k}I_1(2\beta\gamma_k)\right) \qquad (46)$$

is non-injective, there are some points $\gamma_1 \ne \gamma_2$ such that $\lambda(\gamma_1) = \lambda(\gamma_2)$. However, these points were found to be local maxima of the objective function.

Moreover, as we only have one inequality constraint, the point $\boldsymbol{\gamma}_M$ is a *regular point* of all the constraints [9], so the first-order necessary conditions for optimality are met.

Regarding the second order conditions, some of the points in $\mathcal{O}$ can be local maxima, whereas others are local minima. As we are optimizing over the whole set $\mathcal{O}$, it is expected that the solution will lead to a global optimum.

C.2 Properties of the Bessel functions

$$I_v(t) = I_{v-2}(t) - \frac{2(v-1)}{t}I_{v-1}(t), \qquad (47)$$

$$I_v(t) = \frac{2(v-1)}{t}I_{v+1}(t) + I_{v+2}(t), \qquad (48)$$

$$I_1^2(t) > I_0^2(t)I_2^2(t). \qquad (49)$$